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2001 J. Phys. A: Math. Gen. 34 1109

(http://iopscience.iop.org/0305-4470/34/6/304)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 1109–1127 www.iop.org/Journals/ja PII: S0305-4470(01)14010-2

Supersymmetric and shape-invariant generalization for nonresonant and intensity-dependent Jaynes–Cummings systems

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Received 17 May 2000, in final form 2 August 2000

Abstract

A class of shape-invariant bound-state problems which represent transitions in a two-level system introduced earlier are generalized to include arbitrary energy splittings between the two levels as well as intensity-dependent interactions. We show that the coupled-channel Hamiltonians obtained correspond to the generalizations of the nonresonant and intensity-dependent Jaynes–Cummings Hamiltonians, widely used in quantized theories of lasers. In this general context, we determine the eigenstates, eigenvalues, the time evolution matrix and the population inversion matrix factor.

PACS numbers: 0365F, 0370, 0530, 1130P

1. Introduction

The integrability condition called shape invariance originates in supersymmetric quantum mechanics [1, 2]. The separable positive-definite Hamiltonian $\hat{H}_1 = \hat{A}^\dagger \hat{A}$ is called shape-invariant if the condition

$$\hat{A}(a_1)\hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2)\hat{A}(a_2) + R(a_1) \qquad (1.1)$$

is satisfied [3]. In this equation a_1 and a_2 represent parameters of the Hamiltonian. The parameter a_2 is a function of a_1 and the remainder $R(a_1)$ is independent of the dynamical variables such as position and momentum. Even though not all exactly solvable problems are shape invariant [4], shape invariance, especially in its algebraic formulation [5–7], has proven to be a powerful technique to study exactly solvable systems.

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In a previous paper [8] we used shape invariance to calculate the energy eigenvalues and eigenfunctions for the Hamiltonian

$$\hat{\boldsymbol{H}} = \hat{A}^\dagger \hat{A} + \tfrac{1}{2}[\hat{A}, \hat{A}^\dagger](\hat{\sigma}_3 + 1) + \sqrt{\hbar\Omega}(\hat{\sigma}_+ \hat{A} + \hat{\sigma}_- \hat{A}^\dagger) \tag{1.2}$$

where

$$\hat{\sigma}_\pm = \tfrac{1}{2}(\hat{\sigma}_1 \pm \mathrm{i}\hat{\sigma}_2) \tag{1.3}$$

and $\hat{\sigma}_i$, with $i = 1, 2$, and 3, are the Pauli matrices.

This is a generalization of the Jaynes–Cummings Hamiltonian [9]. A different, but related, problem was considered in [10]. Our goal in this paper is to study further generalizations of the Jaynes–Cummings Hamiltonian, first by introducing a term proportional to σ_3 with an arbitrary coefficient (the so-called nonresonant limit) and then by taking into account the dependence of the coupling on the intensity of the field (the so-called intensity-dependent nonresonant limit). In addition to the energy levels we study the time evolution and the population inversion matrix factor.

Introducing the similarity transformation that replaces a_1 with a_2 in a given operator

$$\hat{T}(a_1)\, \hat{O}(a_1)\, \hat{T}^\dagger(a_1) = \hat{O}(a_2) \tag{1.4}$$

and the operators

$$\hat{B}_+ = \hat{A}^\dagger(a_1)\hat{T}(a_1) \tag{1.5}$$

$$\hat{B}_- = \hat{B}_+^\dagger = \hat{T}^\dagger(a_1)\hat{A}(a_1) \tag{1.6}$$

the condition of equation (1.1) can be written as a commutator [5]

$$[\hat{B}_-, \hat{B}_+] = \hat{T}^\dagger(a_1)R(a_1)\hat{T}(a_1) \equiv R(a_0) \tag{1.7}$$

where we used the identity

$$R(a_n) = \hat{T}(a_1)\, R(a_{n-1})\, \hat{T}^\dagger(a_1) \tag{1.8}$$

valid for any n. The ground state of the Hamiltonian $\hat{H}_1 = \hat{A}^\dagger \hat{A} = \hat{B}_+ \hat{B}_-$ satisfies the condition

$$\hat{A}\, |\psi_0\rangle = 0 = \hat{B}_-\, |\psi_0\rangle \tag{1.9}$$

and the unnormalized nth excited state is given by

$$|\psi_n\rangle \sim (\hat{B}_+)^n |\psi_0\rangle \tag{1.10}$$

with the eigenvalue

$$\mathcal{E}_n = \sum_{k=1}^{n} R(a_k). \tag{1.11}$$

We note that the Hamiltonian of equation (1.2) can also be written as

$$\hat{\boldsymbol{H}} = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \hat{\boldsymbol{h}}_\pm \begin{bmatrix} \hat{T}^\dagger & 0 \\ 0 & \pm 1 \end{bmatrix} \tag{1.12}$$

where

$$\hat{\boldsymbol{h}}_\pm = \hat{B}_+ \hat{B}_- + \tfrac{1}{2} R(a_0)(\hat{\sigma}_3 + 1) \pm \sqrt{\hbar\Omega}\, (\hat{\sigma}_+ \hat{B}_- + \hat{\sigma}_- \hat{B}_+). \tag{1.13}$$

2. The generalized nonresonant Jaynes–Cummings Hamiltonian

The standard Jaynes–Cummings model, normally used in quantum optics, idealizes the interaction of matter with electromagnetic radiation by a simple Hamiltonian of a two-level atom coupled to a single bosonic mode [11–16]. This Hamiltonian has a fundamental importance to the field of quantum optics and it is a central ingredient in the quantized description of any optical system involving the interaction between light and atoms. The Jaynes–Cummings Hamiltonian defines a *molecule*, a composite system formed from the coupling of a two-state system and a quantized harmonic oscillator. In this case, its nonresonant expression can be written as

$$\hat{\boldsymbol{H}} = \hat{A}^\dagger \hat{A} + \tfrac{1}{2}[\hat{A}, \hat{A}^\dagger](\hat{\sigma}_3 + 1) + \alpha(\hat{\sigma}_+ \hat{A} + \hat{\sigma}_- \hat{A}^\dagger) + \hbar\Delta\,\hat{\sigma}_3 \tag{2.1}$$

where α is a constant related to the coupling strength and Δ is a constant related to the detuning of the system.

However, the harmonic oscillator system, used in this context, is only the simplest example of a supersymmetric and shape-invariant potential. Our goal here is to generalize that Hamiltonian for all supersymmetric and shape-invariant systems. With this purpose and following [8] we introduce the operator

$$\hat{\boldsymbol{S}} = \hat{\sigma}_+ \hat{A} + \hat{\sigma}_- \hat{A}^\dagger \tag{2.2}$$

where the operators $\hat{A}$ and $\hat{A}^\dagger$ satisfy the shape invariance condition of equation (1.1). Using this definition we can decompose the nonresonant Jaynes–Cummings Hamiltonian in the form

$$\hat{\boldsymbol{H}} = \hat{\boldsymbol{H}}_{\mathrm{o}} + \hat{\boldsymbol{H}}_{\mathrm{int}} \tag{2.3}$$

where

$$\hat{\boldsymbol{H}}_{\mathrm{o}} = \hat{\boldsymbol{S}}^2 \tag{2.4a}$$

$$\hat{\boldsymbol{H}}_{\mathrm{int}} = \alpha\,\hat{\boldsymbol{S}} + \hbar\Delta\,\hat{\sigma}_3. \tag{2.4b}$$

First, we search for the eigenstates of $\hat{\boldsymbol{S}}^2$. In this case it is more convenient to work with its B-operator expression, which can be written as [8]

$$\hat{S}^2 = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \hat{B}_- \hat{B}_+ & 0 \\ 0 & \hat{B}_+ \hat{B}_- \end{bmatrix} \begin{bmatrix} \hat{T}^\dagger & 0 \\ 0 & \pm 1 \end{bmatrix} \equiv \begin{bmatrix} \hat{H}_2 & 0 \\ 0 & \hat{H}_1 \end{bmatrix} \tag{2.5}$$

where $\hat{H}_2 = \hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger$. Note the freedom of sign choice in equation (2.5), which results in two possible decompositions of $\hat{\boldsymbol{S}}^2$. Next, we introduce the states

$$|\Psi^{(\pm)}\rangle = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} C_m^{(\pm)}|m\rangle \\ C_n^{(\pm)}|n\rangle \end{bmatrix} \tag{2.6}$$

where $C_{m,n}^{(\pm)} \equiv C_{m,n}^{(\pm)}[R(a_1), R(a_2), R(a_3), \ldots]$ are auxiliary coefficients, and $|m\rangle$ and $|n\rangle$ are the abbreviated notation for the states $|\psi_m\rangle$ and $|\psi_n\rangle$ of equation (1.10). Using equations (1.7), (2.5) and (2.6), the commutation between $\hat{H}_1$ and a function of $R(a_k)$, and the $\hat{T}$-operator unitarity condition, we get

$$\begin{aligned} \hat{S}^2|\Psi^{(\pm)}\rangle &= \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \hat{B}_+\hat{B}_- + R(a_0) & 0 \\ 0 & \hat{B}_+\hat{B}_- \end{bmatrix} \begin{bmatrix} C_m^{(\pm)}|m\rangle \\ C_n^{(\pm)}|n\rangle \end{bmatrix} \\ &= \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \mathcal{E}_m + R(a_0) & 0 \\ 0 & \mathcal{E}_n \end{bmatrix} \begin{bmatrix} C_m^{(\pm)}|m\rangle \\ C_n^{(\pm)}|n\rangle \end{bmatrix}. \end{aligned} \tag{2.7}$$

Using equations (1.8) and (1.11) we can write

$$\begin{aligned} \hat{T}\,[\mathcal{E}_m + R(a_0)]\,\hat{T}^\dagger &= \hat{T}\,[R(a_1) + R(a_2) + \cdots + R(a_m) + R(a_0)]\,\hat{T}^\dagger \\ &= R(a_2) + R(a_3) + \cdots + R(a_{m+1}) + R(a_1) = \mathcal{E}_{m+1}. \end{aligned} \tag{2.8}$$

Hence the states

$$|\Psi_m^{(\pm)}\rangle = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} C_m^{(\pm)}|m\rangle \\ C_{m+1}^{(\pm)}|m+1\rangle \end{bmatrix} \qquad m = 0, 1, 2, \ldots \tag{2.9}$$

are the normalized eigenstates of the operator $\hat{\boldsymbol{S}}^2$:

$$\hat{\boldsymbol{S}}^2|\Psi_m^{(\pm)}\rangle = \mathcal{E}_{m+1}|\Psi_m^{(\pm)}\rangle. \tag{2.10}$$

We observe that the orthonormality of the wavefunctions imply the following relations among the C:

$$\langle\Psi_m^{(\pm)}|\Psi_m^{(\pm)}\rangle = \left[C_m^{(\pm)}\right]^2 + \left[C_{m+1}^{(\pm)}\right]^2 = 1 \tag{2.11a}$$

$$\langle\Psi_m^{(\mp)}|\Psi_m^{(\pm)}\rangle = C_m^{(\pm)}C_m^{(\mp)} - C_{m+1}^{(\pm)}C_{m+1}^{(\mp)} = 0. \tag{2.11b}$$

Since $\hat{\boldsymbol{S}}^2$ and $\hat{\boldsymbol{H}}_{\text{int}}$ commute then it is possible to find a common set of eigenstates. We can use this fact to determine the eigenvalues of $\hat{\boldsymbol{H}}_{\text{int}}$ and the relations among the C coefficients. For that we need to calculate

$$\hat{\boldsymbol{H}}_{\text{int}}|\Psi_m^{(\pm)}\rangle = \lambda_m^{(\pm)}|\Psi_m^{(\pm)}\rangle \tag{2.12}$$

where $\lambda_m^{(\pm)}$ are the eigenvalues to be determined. Using equations (2.2), (2.4) and (2.9), the last eigenvalue equation can be rewritten in a matrix form as

$$\alpha \begin{bmatrix} \beta & \hat{T}\hat{B}_- \\ \hat{B}_+\hat{T}^\dagger & -\beta \end{bmatrix} \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} C_m^{(\pm)}|m\rangle \\ C_{m+1}^{(\pm)}|m+1\rangle \end{bmatrix} = \lambda_m^{(\pm)} \begin{bmatrix} C_m^{(\pm)}|m\rangle \\ C_{m+1}^{(\pm)}|m+1\rangle \end{bmatrix} \tag{2.13}$$

where $\beta = \hbar\Delta/\alpha$. Since the C coefficients commute with the $\hat{A}$ or $\hat{A}^\dagger$ operators, then the last matrix equation permits us to obtain the following equations:

$$[\alpha\beta - \lambda_m^{(\pm)}](\hat{T}C_m^{(\pm)}\hat{T}^\dagger)\hat{T}|m\rangle \pm \alpha\, C_{m+1}^{(\pm)}\hat{T}\hat{B}_-|m+1\rangle = 0 \tag{2.14a}$$

$$\alpha(\hat{T}C_m^{(\pm)}\hat{T}^\dagger)\hat{B}_+|m\rangle \mp [\alpha\beta + \lambda_m^{(\pm)}]C_{m+1}^{(\pm)}|m+1\rangle = 0. \tag{2.14b}$$

Introducing the operator [7]

$$\hat{Q}^\dagger = (\hat{B}_+\hat{B}_-)^{-1/2}\hat{B}_+ \tag{2.15}$$

one can write the normalized eigenstate of $\hat{H}_1$ as

$$|m\rangle = (\hat{Q}^\dagger)^m|0\rangle \tag{2.16}$$

and with equations (2.15) and (2.16) we can show that [8]

$$\hat{B}_+|m\rangle = \sqrt{\mathcal{E}_{m+1}}|m+1\rangle \tag{2.17a}$$

$$\hat{T}\hat{B}_-|m+1\rangle = \sqrt{\mathcal{E}_{m+1}}\,\hat{T}|m\rangle. \tag{2.17b}$$

Substituting equations (2.17) into (2.14) we have

$$\{[\alpha\beta - \lambda_m^{(\pm)}](\hat{T}C_m^{(\pm)}\hat{T}^\dagger) \pm \alpha\sqrt{\mathcal{E}_{m+1}}\, C_{m+1}^{(\pm)}\}\hat{T}|m\rangle = 0 \tag{2.18a}$$

$$\{\alpha\sqrt{\mathcal{E}_{m+1}}(\hat{T}C_m^{(\pm)}\hat{T}^\dagger) \mp [\alpha\beta + \lambda_m^{(\pm)}]C_{m+1}^{(\pm)}\}|m+1\rangle = 0. \tag{2.18b}$$

From equations (2.18) it follows that

$$\lambda_m^{(\pm)} = \pm\alpha\sqrt{\mathcal{E}_{m+1} + \beta^2} \tag{2.19}$$

and

$$C_{m+1}^{(\pm)} = \left(\frac{\sqrt{\mathcal{E}_{m+1} + \beta^2} \mp \beta}{\sqrt{\mathcal{E}_{m+1}}}\right)(\hat{T}C_m^{(\pm)}\hat{T}^\dagger). \tag{2.20}$$

Equations (2.11) and (2.20) imply that

$$C_{m+1}^{(\pm)} = C_m^{(\mp)} \tag{2.21}$$

and the eigenstates and eigenvalues of the generalized nonresonant Jaynes–Cummings Hamiltonians can be written as

$$E_m^{(\pm)} = \mathcal{E}_{m+1} \pm \sqrt{\alpha^2\,\mathcal{E}_{m+1} + \hbar^2\Delta^2} \tag{2.22}$$

and

$$|\Psi_m^{(\pm)}\rangle = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} C_m^{(\pm)}|m\rangle \\ C_m^{(\mp)}|m+1\rangle \end{bmatrix} \qquad m = 0, 1, 2, \ldots. \tag{2.23}$$

2.1. The resonant limit

From these general results we can verify two important and simple limiting cases. The first one corresponds to the resonant situation, for which $\Delta = 0$ ($\beta = 0$). Using these conditions in equations (2.11), (2.20) and (2.22) we get

$$E_m^{(\pm)} = \mathcal{E}_{m+1} \pm \sqrt{\alpha^2\,\mathcal{E}_{m+1}} \tag{2.24}$$

and

$$C_{m+1}^{(\pm)} = \hat{T} C_m^{(\pm)} \hat{T}^\dagger = C_m^{(\pm)} = \frac{1}{\sqrt{2}}. \tag{2.25}$$

Therefore the Jaynes–Cummings resonant eigenstate is given by

$$|\Psi_m^{(\pm)}\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} |m\rangle \\ |m+1\rangle \end{bmatrix} \qquad m = 0, 1, 2, \ldots. \tag{2.26}$$

These particular results are shown in [8].

2.2. The standard Jaynes–Cummings limit

The second important limit corresponds to the standard Jaynes–Cummings Hamiltonian, related to the harmonic oscillator system. In this limit we have that $\hat{T} = \hat{T}^\dagger \longrightarrow 1$, $\hat{B}_- \longrightarrow \hat{a}$, $\hat{B}_+ \longrightarrow \hat{a}^\dagger$, $\Delta = \omega - \omega_o$ and $\mathcal{E}_{m+1} = (m+1)\hbar\omega$. Using these conditions in equations (2.11), (2.20) and (2.22) we conclude that

$$E_m^{(\pm)} = (m+1)\hbar\omega \pm \sqrt{\alpha^2\hbar\omega(m+1) + \hbar^2(\omega-\omega_o)^2} \tag{2.27}$$

and

$$C_{m+1}^{(\pm)} = \gamma_m^{(\pm)} C_m^{(\pm)} = C_m^{(\mp)} = \frac{1}{\sqrt{1 + (\gamma_m^{(\mp)})^2}} \tag{2.28}$$

where

$$\gamma_m^{(\pm)} = \sqrt{1+\delta_m^2} \mp \delta_m \tag{2.29a}$$

$$\delta_m = \frac{\hbar(\omega-\omega_o)}{\sqrt{(m+1)\alpha^2\hbar\omega}}. \tag{2.29b}$$

Therefore the standard Jaynes–Cummings eigenstate, written in a matrix form, is given by

$$|\Psi_m^{(\pm)}\rangle = \frac{1}{\sqrt{1+(\gamma_m^{(\pm)})^2}} \begin{bmatrix} 1 & 0 \\ 0 & \pm\gamma_m^{(\pm)} \end{bmatrix} \begin{bmatrix} |m\rangle \\ |m+1\rangle \end{bmatrix} \qquad m = 0, 1, 2, \ldots. \tag{2.30}$$

These results are shown in many papers, in particular, in [17].

3. The time evolution of the nonresonant system

To study the time-dependent Schrödinger equation for a Jaynes–Cummings system in a nonresonant situation:

$$i\hbar \frac{\partial}{\partial t}|\Psi(t)\rangle = (\hat{\boldsymbol{H}}_{\mathrm{o}} + \hat{\boldsymbol{H}}_{\mathrm{int}})|\Psi(t)\rangle \tag{3.1}$$

we can write the wavefunction as

$$|\Psi(t)\rangle = \exp(-i\hat{\boldsymbol{H}}_{\mathrm{o}} t/\hbar)\,|\Psi_i(t)\rangle \tag{3.2}$$

and, by substituting this into the Schrödinger equation and taking into account the commutation property between $\hat{\boldsymbol{H}}_{\mathrm{o}}$ and $\hat{\boldsymbol{H}}_{\mathrm{int}}$, we obtain

$$i\hbar \frac{\partial}{\partial t}|\Psi_i(t)\rangle = \hat{\boldsymbol{H}}_{\mathrm{int}}\,|\Psi_i(t)\rangle. \tag{3.3}$$

We introduce the evolution matrix $\hat{\boldsymbol{U}}_i(t,0)$:

$$|\Psi_i(t)\rangle = \hat{\boldsymbol{U}}_i(t,0)\,|\Psi_i(0)\rangle. \tag{3.4}$$

which satisfies the equation

$$i\hbar \frac{\partial}{\partial t}\hat{\boldsymbol{U}}_i(t,0) = \hat{\boldsymbol{H}}_{\mathrm{int}}\,\hat{\boldsymbol{U}}_i(t,0) \tag{3.5}$$

that is, in matrix form, written as

$$i\hbar \begin{bmatrix} \hat{U}'_{11} & \hat{U}'_{12} \\ \hat{U}'_{21} & \hat{U}'_{22} \end{bmatrix} = \alpha \begin{bmatrix} \beta & \hat{T}\hat{B}_- \\ \hat{B}_+\hat{T}^\dagger & -\beta \end{bmatrix} \begin{bmatrix} \hat{U}_{11} & \hat{U}_{12} \\ \hat{U}_{21} & \hat{U}_{22} \end{bmatrix} \tag{3.6}$$

where the primes denote the time derivative. One fast way to diagonalize the evolution matrix differential equation is by differentiating equation (3.5) with respect to time. We find

$$i\hbar \frac{\partial^2}{\partial t^2}\hat{\boldsymbol{U}}_i(t,0) = \hat{\boldsymbol{H}}_{\mathrm{int}} \frac{\partial}{\partial t}\hat{\boldsymbol{U}}_i(t,0) = \frac{1}{i\hbar}\hat{\boldsymbol{H}}^2_{\mathrm{int}}\hat{\boldsymbol{U}}_i(t,0) \tag{3.7}$$

which can be written as

$$\begin{bmatrix} \hat{U}''_{11} & \hat{U}''_{12} \\ \hat{U}''_{21} & \hat{U}''_{22} \end{bmatrix} = -\begin{bmatrix} \hat{\omega}_1 & 0 \\ 0 & \hat{\omega}_2 \end{bmatrix} \begin{bmatrix} \hat{U}_{11} & \hat{U}_{12} \\ \hat{U}_{21} & \hat{U}_{22} \end{bmatrix} \tag{3.8}$$

where

$$\hbar\hat{\omega}_1 = \alpha\sqrt{\hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger + \beta^2} = \sqrt{\alpha^2\,\hat{H}_2 + (\hbar\Delta)^2} \tag{3.9a}$$

$$\hbar\hat{\omega}_2 = \alpha\sqrt{\hat{B}_+\hat{B}_- + \beta^2} = \sqrt{\alpha^2\,\hat{H}_1 + (\hbar\Delta)^2}. \tag{3.9b}$$

Now, since by initial conditions $\hat{\boldsymbol{U}}_i(0,0) = \hat{\boldsymbol{I}}$, then we can write the solution of the evolution matrix differential equation (3.7) as

$$\hat{\boldsymbol{U}}_i(t,0) = \begin{bmatrix} \cos(\hat{\omega}_1 t) & \sin(\hat{\omega}_1 t)\,\hat{C} \\ \sin(\hat{\omega}_2 t)\,\hat{D} & \cos(\hat{\omega}_2 t) \end{bmatrix} \tag{3.10}$$

and the $\hat{C}$ and $\hat{D}$ operators can be determined by the unitarity conditions

$$\hat{\boldsymbol{U}}_i^\dagger(t,0)\,\hat{\boldsymbol{U}}_i(t,0) = \hat{\boldsymbol{U}}_i(t,0)\,\hat{\boldsymbol{U}}_i^\dagger(t,0) = \hat{\boldsymbol{I}}. \tag{3.11}$$

In appendix A we show that the unitarity conditions (3.11) imply

$$\hat{C} = -\hat{D}^\dagger = \frac{i}{(\hat{H}_2)^{1/4}}\sqrt{\hat{T}\hat{B}_-} \tag{3.12a}$$

$$\hat{D} = -\hat{C}^\dagger. \tag{3.12b}$$

Therefore, we can write the final expression of the time evolution matrix of the system as

$$\hat{\boldsymbol{U}}_i(t,0) = \begin{bmatrix} \cos(\hat{\omega}_1 t) & \sin(\hat{\omega}_1 t)\,\hat{C} \\ -\sin(\hat{\omega}_2 t)\,\hat{C}^\dagger & \cos(\hat{\omega}_2 t) \end{bmatrix}. \tag{3.13}$$

For Jaynes–Cummings systems an important physical quantity to see how the system under consideration evolves in time is the population inversion factor [11, 13, 15], defined by

$$\hat{\boldsymbol{W}}(t) \equiv \hat{\sigma}_+(t)\,\hat{\sigma}_-(t) - \hat{\sigma}_-(t)\,\hat{\sigma}_+(t) = \hat{\sigma}_3(t) \tag{3.14}$$

where the time dependence of the operators is related to the Heisenberg picture. In this case, the time evolution of the population inversion factor will be given by

$$\frac{\mathrm{d}\hat{\sigma}_3(t)}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}\hat{\boldsymbol{U}}_i^\dagger(t,0)[\hat{\sigma}_3, \hat{\boldsymbol{H}}]\hat{\boldsymbol{U}}_i(t,0) \tag{3.15}$$

and since we have

$$[\hat{\sigma}_3, \hat{\boldsymbol{H}}] = \alpha[\hat{\sigma}_3, \hat{\boldsymbol{S}}] = -2\alpha\,\hat{\boldsymbol{S}}\,\hat{\sigma}_3 \tag{3.16}$$

then equation (3.15) can be written as

$$\frac{\mathrm{d}\hat{\sigma}_3(t)}{\mathrm{d}t} = \frac{2\mathrm{i}\alpha}{\hbar}\,\hat{\boldsymbol{S}}(t)\,\hat{\sigma}_3(t). \tag{3.17}$$

We can obtain a differential equation with constant coefficients for $\hat{\sigma}_3(t)$ by taking the time derivative of equation (3.17):

$$\frac{\mathrm{d}^2\hat{\sigma}_3(t)}{\mathrm{d}t^2} = \frac{2\mathrm{i}\alpha}{\hbar}\left\{\frac{\mathrm{d}\hat{\boldsymbol{S}}(t)}{\mathrm{d}t}\,\hat{\sigma}_3(t) + \hat{\boldsymbol{S}}(t)\,\frac{\mathrm{d}\hat{\sigma}_3(t)}{\mathrm{d}t}\right\}. \tag{3.18}$$

Having in mind that

$$\frac{\mathrm{d}\hat{\boldsymbol{S}}(t)}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}\hat{\boldsymbol{U}}_i^\dagger(t,0)[\hat{\boldsymbol{S}}, \hat{\boldsymbol{H}}]\hat{\boldsymbol{U}}_i(t,0) \tag{3.19}$$

and that

$$[\hat{\boldsymbol{S}}, \hat{\boldsymbol{H}}] = \alpha\beta[\hat{\boldsymbol{S}}, \hat{\sigma}_3] = 2\alpha\beta\,\hat{\boldsymbol{S}}\,\hat{\sigma}_3 \tag{3.20}$$

we conclude that

$$\frac{\mathrm{d}\hat{\boldsymbol{S}}(t)}{\mathrm{d}t} = -\frac{2\mathrm{i}\alpha\beta}{\hbar}\,\hat{\boldsymbol{S}}(t)\,\hat{\sigma}_3(t). \tag{3.21}$$

Using equations (3.17) and (3.21) in (3.18) we obtain

$$\frac{\mathrm{d}^2\hat{\sigma}_3(t)}{\mathrm{d}t^2} + \hat{\boldsymbol{\Theta}}^2\,\hat{\sigma}_3(t) = \hat{\boldsymbol{F}}(t) \tag{3.22}$$

where

$$\hat{\boldsymbol{\Theta}}^2 = \frac{4\alpha^2}{\hbar^2}\,\hat{\boldsymbol{S}}^2 \tag{3.23a}$$

$$\hat{\boldsymbol{F}}(t) = \frac{4\alpha^2\beta}{\hbar^2}\,\hat{\boldsymbol{U}}_i^\dagger(t,0)\,\hat{\boldsymbol{S}}\,\hat{\boldsymbol{U}}_i(t,0). \tag{3.23b}$$

Equation (3.22) corresponds to a non-homogeneous linear differential equation for $\hat{\sigma}_3(t)$ with constant coefficients since $\hat{\boldsymbol{S}}^2$ and $\hat{\boldsymbol{H}}$ commute and, therefore, $\hat{\boldsymbol{\Theta}}$ is a constant of the motion. The general solution of this differential equation can be written as

$$\hat{\sigma}_3(t) = \hat{\sigma}^H(t) + \hat{\sigma}^P(t) \tag{3.24}$$

and each matrix element of the homogeneous solution satisfies the differential equation

$$\frac{\mathrm{d}^2\hat{\sigma}^H_{jk}(t)}{\mathrm{d}t^2} + \hat{\nu}_j^2\,\hat{\sigma}^H_{jk}(t) = 0 \qquad j, k = 1, \text{ or } 2 \tag{3.25}$$

with

$$\hbar\hat{\nu}_1 = 2\alpha\sqrt{\hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger} = 2\sqrt{\alpha^2\,\hat{H}_2} \tag{3.26a}$$

$$\hbar\hat{\nu}_2 = 2\alpha\sqrt{\hat{B}_+\hat{B}_-} = 2\sqrt{\alpha^2\,\hat{H}_1}. \tag{3.26b}$$

The solution of equation (3.25) is given by

$$\hat{\sigma}^H_{jk}(t) = \hat{y}_j(t)\,\hat{c}_{jk} + \hat{z}_j(t)\,\hat{d}_{jk} \tag{3.27}$$

where

$$\hat{y}_j(t) = \cos(\hat{\nu}_j t) \tag{3.28a}$$

$$\hat{z}_j(t) = \sin(\hat{\nu}_j t) \tag{3.28b}$$

and the coefficients $\hat{c}_{jk}$ and $\hat{d}_{jk}$ can be determined by the initial conditions.

The matrix elements of the particular solution of the $\hat{\sigma}_3(t)$ differential equation need to satisfy

$$\frac{\mathrm{d}^2\hat{\sigma}^P_{jk}(t)}{\mathrm{d}t^2} + \hat{\nu}_j^2\,\hat{\sigma}^P_{jk}(t) = \hat{F}_{jk}(t) \qquad j, k = 1, \text{ or } 2 \tag{3.29}$$

and they can be obtained by the variation of parameter or by Green function methods, giving

$$\hat{\sigma}^P_{jk}(t) = \hat{\nu}_j^{-1}\left\{\hat{z}_j(t)\int_0^t \mathrm{d}\xi\;\hat{y}_j(\xi)\,\hat{F}_{jk}(\xi) - \hat{y}_j(t)\int_0^t \mathrm{d}\xi\;\hat{z}_j(\xi)\,\hat{F}_{jk}(\xi)\right\} \tag{3.30}$$

where we used that the Wronskian of the system of solutions $\hat{y}_j(t)$ and $\hat{z}_j(t)$ is given by $\hat{\nu}_j$.

After we determine the elements of the $\hat{\boldsymbol{F}}(t)$ matrix, it is necessary to resolve the integrals in equation (3.30) to obtain the explicit expression of the particular solution. In appendix B we show that, using equations (2.2), (3.13) and (3.23), it is possible to conclude that these matrix elements can be written as

$$\hat{\sigma}^P_{11}(t) = \mathrm{i}\frac{\gamma}{2}\hat{\nu}_1^{-1}\sqrt{\hat{T}\hat{B}_-}\{\hat{z}_2(t)\,\mathcal{G}^{(+)}_{CS}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1) - \hat{y}_2(t)\,\mathcal{G}^{(+)}_{SS}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1)\}\hat{H}_2^{1/4}$$
$$+\mathrm{i}\frac{\gamma}{2}\hat{\nu}_1^{-1}\hat{H}_2^{1/4}\{\hat{z}_1(t)\,\mathcal{G}^{(-)}_{SC}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2) - \hat{y}_1(t)\,\mathcal{G}^{(-)}_{CC}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2)\}\sqrt{\hat{B}_+\hat{T}^\dagger} \tag{3.31a}$$

$$\hat{\sigma}^P_{12}(t) = \frac{\gamma}{2}\hat{\nu}_1^{-1}\sqrt{\hat{T}\hat{B}_-}\{\hat{z}_2(t)\,\mathcal{G}^{(+)}_{CC}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1) - \hat{y}_2(t)\,\mathcal{G}^{(+)}_{SC}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1)\}\sqrt{\hat{T}\hat{B}_-}$$
$$+\frac{\gamma}{2}\hat{\nu}_1^{-1}\hat{H}_2^{1/4}\{\hat{z}_1(t)\,\mathcal{G}^{(-)}_{SS}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2) + \hat{y}_1(t)\,\mathcal{G}^{(-)}_{CS}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2)\}\hat{H}_1^{1/4} \tag{3.31b}$$

$$\hat{\sigma}^P_{21}(t) = \frac{\gamma}{2}\hat{\nu}_2^{-1}\sqrt{\hat{B}_+\hat{T}^\dagger}\{\hat{z}_1(t)\,\mathcal{G}^{(+)}_{CC}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2) - \hat{y}_1(t)\,\mathcal{G}^{(+)}_{SC}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2)\}\sqrt{\hat{B}_+\hat{T}^\dagger}$$
$$+\frac{\gamma}{2}\hat{\nu}_2^{-1}\hat{H}_1^{1/4}\{\hat{z}_2(t)\,\mathcal{G}^{(-)}_{SS}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1) - \hat{y}_2(t)\,\mathcal{G}^{(-)}_{CS}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1)\}\hat{H}_2^{1/4} \tag{3.31c}$$

$$\hat{\sigma}^P_{22}(t) = \mathrm{i}\frac{\gamma}{2}\hat{\nu}_2^{-1}\sqrt{\hat{B}_+\hat{T}^\dagger}\{\hat{z}_1(t)\,\mathcal{G}^{(+)}_{CS}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2) - \hat{y}_1(t)\,\mathcal{G}^{(+)}_{SS}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2)\}\hat{H}_1^{1/4}$$
$$+\mathrm{i}\frac{\gamma}{2}\hat{\nu}_2^{-1}\hat{H}_1^{1/4}\{\hat{z}_2(t)\,\mathcal{G}^{(-)}_{SC}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1) + \hat{y}_2(t)\,\mathcal{G}^{(-)}_{CC}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1)\}\sqrt{\hat{T}\hat{B}_-} \tag{3.31d}$$

where $\gamma = 4\alpha^2\beta/\hbar^2$, and the auxiliary functions are given by

$$\mathcal{G}^{(\pm)}_{XY}(t;\, \hat{p}, \hat{q}, \hat{r}) = \mathcal{F}_{XY}(t;\, \hat{p} - \hat{q}, \hat{r}) \pm \mathcal{F}_{XY}(t;\, \hat{p} + \hat{q}, \hat{r}) \qquad X, Y = C \text{ or } S \tag{3.32}$$

with

$$\mathcal{F}_{CC}(t;\, \hat{x}, \hat{w}) \equiv \int_0^t \mathrm{d}\xi\, \cos(\hat{x}\xi)\, \cos(\hat{w}\xi) = \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{\hat{x}^{2m}\hat{w}^{2n}}{(2m)!\,(2n)!} \frac{t^{2m+2n+1}}{(2m+2n+1)} \tag{3.33a}$$

$$\mathcal{F}_{CS}(t;\, \hat{x}, \hat{w}) \equiv \int_0^t \mathrm{d}\xi\, \cos(\hat{x}\xi)\, \sin(\hat{w}\xi) = \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{\hat{x}^{2m}\hat{w}^{2n+1}}{(2m)!\,(2n+1)!} \frac{t^{2m+2n+2}}{(2m+2n+2)} \tag{3.33b}$$

$$\mathcal{F}_{SC}(t;\, \hat{x}, \hat{w}) \equiv \int_0^t \mathrm{d}\xi\, \sin(\hat{x}\xi)\, \cos(\hat{w}\xi) = \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{\hat{x}^{2m+1}\hat{w}^{2n}}{(2m+1)!\,(2n)!} \frac{t^{2m+2n+2}}{(2m+2n+2)} \tag{3.33c}$$

$$\mathcal{F}_{SS}(t;\, \hat{x}, \hat{w}) \equiv \int_0^t \mathrm{d}\xi\, \sin(\hat{x}\xi)\, \sin(\hat{w}\xi) = \sum_{m,n=0}^{\infty} (-1)^{m+n} \frac{\hat{x}^{2m+1}\hat{w}^{2n+1}}{(2m+1)!\,(2n+1)!} \frac{t^{2m+2n+3}}{(2m+2n+3)}. \tag{3.33d}$$

With these results for the particular solution we can conclude that

$$\hat{\sigma}^P_{ij}(0) = 0 = \frac{\mathrm{d}\hat{\sigma}^P_{ij}(0)}{\mathrm{d}t}. \tag{3.34}$$

Now, using equations (3.17), (3.24), (3.27), (3.34) and the initial conditions, we have

$$[\hat{\sigma}_3(0)]_{ij} = \hat{c}_{ij} \tag{3.35a}$$

$$\left[\frac{\mathrm{d}\hat{\sigma}_3(0)}{\mathrm{d}t}\right]_{ij} = \frac{2\mathrm{i}\alpha}{\hbar}\,[\hat{\boldsymbol{S}}(0)\,\hat{\sigma}_3(0)]_{ij} = \hat{\nu}_i\,\hat{d}_{ij}. \tag{3.35b}$$

Therefore, the final expression for the elements of the population inversion matrix of the system can be written as

$$[\hat{\sigma}_3(t)]_{ij} = \cos(\hat{\nu}_i t)\,[\hat{\sigma}_3(0)]_{ij} + \frac{2\mathrm{i}\alpha}{\hbar}\,\sin(\hat{\nu}_i t)\,\hat{\nu}_i^{-1}[\hat{\boldsymbol{S}}(0)\,\hat{\sigma}_3(0)]_{ij} + \hat{\sigma}^P_{ij}(t). \tag{3.36}$$

Again, using these final results we can verify two important and simple limit cases.

3.1. *The resonant limit*

The first one corresponds to the resonant situation ($\Delta = 0$). Equations (3.9), (3.13), (3.26) and (3.31) allow us to conclude that, in this case, the evolution matrix of the system is given by

$$\hat{\boldsymbol{U}}_i(t, 0) = \begin{bmatrix} \cos\left(\frac{1}{2}\hat{\nu}_1 t\right) & \sin\left(\frac{1}{2}\hat{\nu}_1 t\right)\hat{C} \\ -\sin\left(\frac{1}{2}\hat{\nu}_2 t\right)\hat{C}^\dagger & \cos\left(\frac{1}{2}\hat{\nu}_2 t\right) \end{bmatrix} \tag{3.37}$$

and the elements of the population inversion of the system are

$$[\hat{\sigma}_3(t)]_{ij} = \cos(\hat{\nu}_i t)\,[\hat{\sigma}_3(0)]_{ij} + \frac{2\mathrm{i}\alpha}{\hbar}\,\sin(\hat{\nu}_i t)\,\hat{\nu}_i^{-1}[\hat{\boldsymbol{S}}(0)\,\hat{\sigma}_3(0)]_{ij}. \tag{3.38}$$

3.2. The standard Jaynes–Cummings limit

This second important limit corresponds to the case of the harmonic oscillator system, and in this limit we have that $\hat{T} = \hat{T}^\dagger \longrightarrow 1$, $\hat{B}_- \longrightarrow \hat{a}$, $\hat{B}_+ \longrightarrow \hat{a}^\dagger$ and $[\hat{a}, \hat{a}^\dagger] = \hbar\omega$. With these conditions the operators $\hat{\omega}_1$ and $\hat{\omega}_2$ commute, and this fact permits us to evaluate the integrals related to the particular solution of the population inversion elements using trigonometric product relations. Using that and the expressions obtained in appendix B, after a considerable amount of algebra and trigonometric product relations we can show that it is possible to write the expressions for the $\hat{\sigma}_{ij}^P(t)$ matrix elements as

$$\hat{\sigma}_{11}^P(t) = \mathrm{i}\frac{\gamma}{2}\hat{\nu}_1^{-1}\sqrt{\hat{a}}\{\mathcal{K}_S(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_2) - \mathcal{K}_S(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_2)\}(\hat{a}\hat{a}^\dagger)^{1/4}$$
$$-\mathrm{i}\frac{\gamma}{2}\hat{\nu}_1^{-1}(\hat{a}\hat{a}^\dagger)^{1/4}\{\mathcal{K}_S(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_1) - \mathcal{K}_S(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_1)\}\sqrt{\hat{a}^\dagger} \qquad (3.39a)$$

$$\hat{\sigma}_{12}^P(t) = \frac{\gamma}{2}\hat{\nu}_1^{-1}\sqrt{\hat{a}}\{\mathcal{K}_C(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_2) - \mathcal{K}_C(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_2)\}\sqrt{\hat{a}}$$
$$-\frac{\gamma}{2}\hat{\nu}_1^{-1}(\hat{a}\hat{a}^\dagger)^{1/4}\{\mathcal{K}_C(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_1) - \mathcal{K}_C(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_1)\}(\hat{a}^\dagger\hat{a})^{1/4} \qquad (3.39b)$$

$$\hat{\sigma}_{21}^P(t) = \frac{\gamma}{2}\hat{\nu}_2^{-1}\sqrt{\hat{a}^\dagger}\{\mathcal{K}_C(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_1) + \mathcal{K}_C(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_1)\}\sqrt{\hat{a}^\dagger}$$
$$-\frac{\gamma}{2}\hat{\nu}_2^{-1}(\hat{a}^\dagger\hat{a})^{1/4}\{\mathcal{K}_C(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_2) - \mathcal{K}_C(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_2)\}(\hat{a}\hat{a}^\dagger)^{1/4} \qquad (3.39c)$$

$$\hat{\sigma}_{22}^P(t) = \mathrm{i}\frac{\gamma}{2}\hat{\nu}_2^{-1}\sqrt{\hat{a}^\dagger}\{\mathcal{K}_S(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_1) + \mathcal{K}_S(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_1)\}(\hat{a}^\dagger\hat{a})^{1/4}$$
$$-\mathrm{i}\frac{\gamma}{2}\hat{\nu}_2^{-1}(\hat{a}^\dagger\hat{a})^{1/4}\{\mathcal{K}_S(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_2) + \mathcal{K}_S(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_2)\}\sqrt{\hat{a}} \qquad (3.39d)$$

where, now, the auxiliary functions are given by

$$\mathcal{K}_S(t;\hat{p},\hat{q},\hat{r}) = \frac{\hat{r}\,\sin[(\hat{p}+\hat{q})t] - (\hat{p}+\hat{q})\,\sin(\hat{r}t)}{\hat{r}^2 - (\hat{p}+\hat{q})^2} \qquad (3.40a)$$

$$\mathcal{K}_C(t;\hat{p},\hat{q},\hat{r}) = \frac{\hat{r}\,\cos[(\hat{p}+\hat{q})t] - \hat{r}\,\cos(\hat{r}t)}{\hat{r}^2 - (\hat{p}+\hat{q})^2}. \qquad (3.40b)$$

Considering the expressions above we may easily verify that the particular solution for the population inversion factor must still satisfy the initial conditions (3.34). Therefore, in this case the final expression for the population inversion factor has the same form given by equation (3.36), with

$$\hbar\hat{\nu}_1 = 2\alpha\sqrt{\hat{a}\hat{a}^\dagger} \qquad \hbar\hat{\nu}_2 = 2\alpha\sqrt{\hat{a}^\dagger\hat{a}} \qquad (3.41a)$$

$$\hbar\hat{\omega}_1 = \alpha\sqrt{\hat{a}\hat{a}^\dagger + \beta^2} \qquad \hbar\hat{\omega}_2 = \alpha\sqrt{\hat{a}^\dagger\hat{a} + \beta^2}. \qquad (3.41b)$$

4. The generalized intensity-dependent nonresonant Jaynes–Cummings Hamiltonian

A variant of the Jaynes–Cummings model takes the coupling between matter and the radiation to depend on the intensity of the electromagnetic field [13, 15, 16, 18]. This model has great relevance since this kind of interaction means effectively that the coupling is proportional to the amplitude of the field, which is a very simple case of a nonlinear interaction corresponding to a more realistic physical situation. The results of this model can also give insight into the behaviour of other quantum systems with strong nonlinear interactions. In this section we generalize the standard intensity-dependent nonresonant Jaynes–Cummings model to a shape-invariant one.

The expression for the intensity-dependent nonresonant Jaynes–Cummings Hamiltonian can be written as

$$\hat{\boldsymbol{H}} = \hat{A}^\dagger \hat{A} + \tfrac{1}{2}[\hat{A}, \hat{A}^\dagger](\hat{\sigma}_3 + 1) + \alpha\left(\hat{\sigma}_+ \hat{A}\sqrt{\hat{A}^\dagger \hat{A}} + \hat{\sigma}_- \sqrt{\hat{A}^\dagger \hat{A}}\,\hat{A}^\dagger\right) + \hbar\Delta\,\hat{\sigma}_3. \tag{4.1}$$

Note that here the constant α is dimensionless. To generalize the Hamiltonian (4.1) for all supersymmetric and shape-invariant systems, we can use the operator $\hat{\boldsymbol{S}}$, given by equation (2.2), and further introduce the operator

$$\hat{\boldsymbol{S}}_i = \hat{\sigma}_+ \hat{A}\sqrt{\hat{A}^\dagger \hat{A}} + \hat{\sigma}_- \sqrt{\hat{A}^\dagger \hat{A}}\,\hat{A}^\dagger. \tag{4.2}$$

Again, the operators $\hat{A}$ and $\hat{A}^\dagger$ satisfy the shape invariance condition, equation (1.1). Using operators $\hat{\boldsymbol{S}}$ and $\hat{\boldsymbol{S}}_i$ we can decompose the Jaynes–Cummings Hamiltonian of equation (4.1) in the form

$$\hat{\boldsymbol{H}} = \hat{\boldsymbol{H}}_{\rm o} + \hat{\boldsymbol{H}}_{\rm int} \tag{4.3}$$

where

$$\hat{\boldsymbol{H}}_{\rm o} = \hat{\boldsymbol{S}}^2 \tag{4.4a}$$

$$\hat{\boldsymbol{H}}_{\rm int} = \alpha\,\hat{\boldsymbol{S}}_i + \hbar\Delta\,\hat{\sigma}_3. \tag{4.4b}$$

In this case, $\hat{\boldsymbol{H}}_{\rm int}$ can be written as

$$\hat{\boldsymbol{H}}_{\rm int} = \alpha\begin{bmatrix} \beta & \hat{T}\hat{B}_-\sqrt{\hat{B}_+\hat{B}_-} \\ \sqrt{\hat{B}_+\hat{B}_-}\,\hat{B}_+\hat{T}^\dagger & -\beta \end{bmatrix} = \alpha\begin{bmatrix} \beta & \hat{T}\hat{B}_-\sqrt{\hat{H}_1} \\ \sqrt{\hat{H}_1}\,\hat{B}_+\hat{T}^\dagger & -\beta \end{bmatrix}. \tag{4.5}$$

Here we can follow the same development as in section 2, with the same notation. Hence, using equations (2.2), (2.9), (4.2) and (4.4), the eigenvalue equation

$$\hat{\boldsymbol{H}}_{\rm int}|\Psi_m^{(\pm)}\rangle = \lambda_m^{(\pm)}|\Psi_m^{(\pm)}\rangle \tag{4.6}$$

can be written in matrix form as

$$\alpha\begin{bmatrix} \beta & \hat{T}\hat{B}_-\sqrt{\hat{H}_1} \\ \sqrt{\hat{H}_1}\hat{B}_+\hat{T}^\dagger & -\beta \end{bmatrix}\begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix}\begin{bmatrix} C_m^{(\pm)}|m\rangle \\ C_{m+1}^{(\pm)}|m+1\rangle \end{bmatrix} = \lambda_m^{(\pm)}\begin{bmatrix} C_m^{(\pm)}|m\rangle \\ C_{m+1}^{(\pm)}|m+1\rangle \end{bmatrix}. \tag{4.7}$$

Again, since the C coefficients commute with the $\hat{A}$ or $\hat{A}^\dagger$ operators, then the last matrix equation permits us to obtain the following equations:

$$[\alpha\beta - \lambda_m^{(\pm)}](\hat{T}C_m^{(\pm)}\hat{T}^\dagger)\hat{T}|m\rangle \pm \alpha\, C_{m+1}^{(\pm)}\hat{T}\hat{B}_-\sqrt{\hat{H}_1}|m+1\rangle = 0 \tag{4.8a}$$

$$\alpha(\hat{T}C_m^{(\pm)}\hat{T}^\dagger)\sqrt{\hat{H}_1}\,\hat{B}_+|m\rangle \mp [\alpha\beta + \lambda_m^{(\pm)}]C_{m+1}^{(\pm)}|m+1\rangle = 0. \tag{4.8b}$$

Now, using equations (2.15)–(2.17) we have

$$\begin{aligned}\hat{T}\hat{B}_-\sqrt{\hat{H}_1}|m+1\rangle &= \hat{T}\hat{B}_-\sqrt{\mathcal{E}_{m+1}}|m+1\rangle \\ &= \sqrt{\mathcal{E}_{m+1}}\,\hat{T}\hat{B}_-|m+1\rangle \\ &= \mathcal{E}_{m+1}\,\hat{T}|m\rangle\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\sqrt{\hat{H}_1}\,\hat{B}_+|m\rangle &= \sqrt{\hat{H}_1}\,\sqrt{\mathcal{E}_{m+1}}|m+1\rangle \\ &= \sqrt{\mathcal{E}_{m+1}}\,\sqrt{\hat{H}_1}|m+1\rangle \\ &= \mathcal{E}_{m+1}|m+1\rangle.\end{aligned} \tag{4.10}$$

Using equations (4.9) and (4.10), then equations (4.8) take the form

$$\{[\alpha\beta - \lambda_m^{(\pm)}](\hat{T} C_m^{(\pm)} \hat{T}^\dagger) \pm \alpha\, \mathcal{E}_{m+1}\, C_{m+1}^{(\pm)}\}\hat{T}|m\rangle = 0 \tag{4.11a}$$

$$\{\alpha\, \mathcal{E}_{m+1}(\hat{T} C_m^{(\pm)} \hat{T}^\dagger) \mp [\alpha\beta + \lambda_m^{(\pm)}] C_{m+1}^{(\pm)}\}|m+1\rangle = 0. \tag{4.11b}$$

From equations (4.11) it follows that

$$\lambda_m^{(\pm)} = \pm\alpha\sqrt{\mathcal{E}_{m+1}^2 + \beta^2} \tag{4.12}$$

and

$$C_{m+1}^{(\pm)} = \left(\frac{\sqrt{\mathcal{E}_{m+1}^2 + \beta^2} \mp \beta}{\mathcal{E}_{m+1}}\right)(\hat{T} C_m^{(\pm)} \hat{T}^\dagger). \tag{4.13}$$

Equations (2.11) and (4.13) imply that

$$C_{m+1}^{(\pm)} = C_m^{(\mp)} \tag{4.14}$$

and the eigenstates and eigenvalues of the generalized intensity-dependent nonresonant Jaynes–Cummings Hamiltonian can be written as

$$E_m^{(\pm)} = \mathcal{E}_{m+1} \pm \sqrt{\alpha^2\, \mathcal{E}_{m+1}^2 + \hbar^2 \Delta^2} \tag{4.15}$$

and

$$|\Psi_m^{(\pm)}\rangle = \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} C_m^{(\pm)}|m\rangle \\ C_m^{(\mp)}|m+1\rangle \end{bmatrix} \qquad m = 0, 1, 2, \ldots. \tag{4.16}$$

4.1. The intensity-dependent resonant limit

From these general results we can again verify the two simple limiting cases. The first one, corresponding to the resonant situation, is for $\Delta = 0$ ($\beta = 0$). Using these conditions in equations (2.11), (4.13) and (4.15) we can promptly conclude that

$$E_m^{(\pm)} = (1 \pm \alpha)\, \mathcal{E}_{m+1} \tag{4.17}$$

and

$$C_{m+1}^{(\pm)} = \hat{T} C_m^{(\pm)} \hat{T}^\dagger = C_m^{(\pm)} = \frac{1}{\sqrt{2}}. \tag{4.18}$$

Therefore the intensity-dependent resonant Jaynes–Cummings eigenstate is given by

$$|\Psi_m^{(\pm)}\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{T} & 0 \\ 0 & \pm 1 \end{bmatrix} \begin{bmatrix} |m\rangle \\ |m+1\rangle \end{bmatrix} \qquad m = 0, 1, 2, \ldots. \tag{4.19}$$

If we compare this last particular result with that found in [8], we conclude that the intensity-dependent and intensity-independent generalized Jaynes–Cummings Hamiltonians have the same eigenstates in the resonant situation.

4.2. The standard intensity-dependent Jaynes–Cummings limit

The second limit, corresponding to the standard intensity-dependent Jaynes–Cummings case, is related to the harmonic oscillator system. In this limit we have that $\hat{T} = \hat{T}^\dagger \longrightarrow 1$, $\hat{B}_- \longrightarrow \hat{a}$, $\hat{B}_+ \longrightarrow \hat{a}^\dagger$, $\Delta = \omega - \omega_o$ and $\mathcal{E}_{m+1} = (m+1)\hbar\omega$. Using these conditions in equations (2.11), (4.13) and (4.15) we can promptly conclude that

$$E_m^{(\pm)} = (m+1)\hbar\omega \pm \hbar\sqrt{\alpha^2\omega^2(m+1)^2 + (\omega - \omega_o)^2} \tag{4.20}$$

and

$$C_{m+1}^{(\pm)} = \gamma_m^{(\pm)} C_m^{(\pm)} = C_m^{(\mp)} = \frac{1}{\sqrt{1 + (\gamma_m^{(\mp)})^2}} \tag{4.21}$$

where

$$\gamma_m^{(\pm)} = \sqrt{1 + \delta_m^2} \mp \delta_m \tag{4.22a}$$

$$\delta_m = \frac{\omega - \omega_o}{\alpha\omega(m+1)}. \tag{4.22b}$$

Therefore the standard intensity-dependent Jaynes–Cummings eigenstate, written in a matrix form, is given by

$$|\Psi_m^{(\pm)}\rangle = \frac{1}{\sqrt{1 + (\gamma_m^{(\pm)})^2}} \begin{bmatrix} 1 & 0 \\ 0 & \pm\gamma_m^{(\pm)} \end{bmatrix} \begin{bmatrix} |m\rangle \\ |m+1\rangle \end{bmatrix} \qquad m = 0, 1, 2, \ldots. \tag{4.23}$$

5. Time evolution of the intensity-dependent nonresonant system

To resolve the time-dependent Schrödinger equation for intensity-dependent nonresonant Jaynes–Cummings systems:

$$\mathrm{i}\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = (\hat{\boldsymbol{H}}_o + \hat{\boldsymbol{H}}_{\text{int}}) |\Psi(t)\rangle \tag{5.1}$$

we can again write the state $|\Psi(t)\rangle$ as it is given by equation (3.2). Then using equations (3.3)–(3.5), we can write the matrix equation

$$\mathrm{i}\hbar \begin{bmatrix} \hat{U}_{11}' & \hat{U}_{12}' \\ \hat{U}_{21}' & \hat{U}_{22}' \end{bmatrix} = \alpha \begin{bmatrix} \beta & \hat{T}\hat{B}_- \sqrt{\hat{H}_1} \\ \sqrt{\hat{H}_1}\, \hat{B}_+ \hat{T}^\dagger & -\beta \end{bmatrix} \begin{bmatrix} \hat{U}_{11} & \hat{U}_{12} \\ \hat{U}_{21} & \hat{U}_{22} \end{bmatrix}. \tag{5.2}$$

To diagonalize this evolution matrix differential equation we can differentiate equation (3.5) with respect to time. After that, using equation (3.5), we find

$$\mathrm{i}\hbar \frac{\partial^2}{\partial t^2} \hat{\boldsymbol{U}}_i(t,0) = \hat{\boldsymbol{H}}_{\text{int}} \frac{\partial}{\partial t} \hat{\boldsymbol{U}}_i(t,0) = \frac{1}{\mathrm{i}\hbar} \hat{\boldsymbol{H}}_{\text{int}}^2 \hat{\boldsymbol{U}}_i(t,0) \tag{5.3}$$

which can be written as

$$\begin{bmatrix} \hat{U}_{11}'' & \hat{U}_{12}'' \\ \hat{U}_{21}'' & \hat{U}_{22}'' \end{bmatrix} = - \begin{bmatrix} \hat{\omega}_1 & 0 \\ 0 & \hat{\omega}_2 \end{bmatrix} \begin{bmatrix} \hat{U}_{11} & \hat{U}_{12} \\ \hat{U}_{21} & \hat{U}_{22} \end{bmatrix} \tag{5.4}$$

where

$$\hbar\hat{\omega}_1 = \alpha\sqrt{(\hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger)^2 + \beta^2} = \sqrt{\alpha^2\,\hat{H}_2^2 + (\hbar\Delta)^2} \tag{5.5a}$$

$$\hbar\hat{\omega}_2 = \alpha\sqrt{(\hat{B}_+\hat{B}_-)^2 + \beta^2} = \sqrt{\alpha^2\,\hat{H}_1^2 + (\hbar\Delta)^2}. \tag{5.5b}$$

Now, using the initial conditions $\hat{\boldsymbol{U}}_i(0,0) = \hat{\boldsymbol{I}}$, we can write the solution of the evolution matrix differential equation (5.3) as

$$\hat{\boldsymbol{U}}_i(t,0) = \begin{bmatrix} \cos(\hat{\omega}_1 t) & \sin(\hat{\omega}_1 t)\,\hat{C} \\ \sin(\hat{\omega}_2 t)\,\hat{D} & \cos(\hat{\omega}_2 t) \end{bmatrix} \tag{5.6}$$

where the $\hat{C}$ and $\hat{D}$ operators can be determined by equation (3.11). Following the same steps used in appendix A, we can conclude that these operators must have the form given by equations (3.12). So, in this case the final expression of the time evolution matrix $\hat{\boldsymbol{U}}_i(t,0)$ is given by equation (3.13) as well.

To obtain the population inversion factor we can again follow the steps from equation (3.14) to (3.30), but replacing the operator $\hat{\boldsymbol{S}}$ by the operator $\hat{\boldsymbol{S}}_i$. Besides that we have

$$\hbar\hat{\nu}_1 = 2\alpha\,\hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger = 2\alpha\,\hat{H}_2 \tag{5.7a}$$

$$\hbar\hat{\nu}_2 = 2\alpha\,\hat{B}_+\hat{B}_- = 2\alpha\,\hat{H}_1 \tag{5.7b}$$

instead of equations (3.26). Here, we can again use the development shown in appendix B, just replacing $\hat{\boldsymbol{S}}$ by $\hat{\boldsymbol{S}}_i$, to obtain the explicit form of the matrix elements for the particular solution of the population inversion factor, given by equation (3.30). So for a shape-invariant intensity-dependent nonresonant Jaynes–Cummings system, these matrix elements are given by

$$\hat{\sigma}_{11}^P(t) = \mathrm{i}\frac{\gamma}{2}\hat{\nu}_1^{-1}\sqrt{\hat{T}\hat{B}_-}\{\hat{z}_2(t)\,\mathcal{G}_{CS}^{(+)}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1) - \hat{y}_2(t)\,\mathcal{G}_{SS}^{(+)}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1)\}\hat{H}_2^{3/4} + \mathrm{i}\frac{\gamma}{2}\hat{\nu}_1^{-1}\hat{H}_2^{3/4}\{\hat{z}_1(t)\,\mathcal{G}_{SC}^{(-)}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2) - \hat{y}_1(t)\,\mathcal{G}_{CC}^{(-)}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2)\}\sqrt{\hat{B}_+\hat{T}^\dagger} \tag{5.8a}$$

$$\hat{\sigma}_{12}^P(t) = \frac{\gamma}{2}\hat{\nu}_1^{-1}\sqrt{\hat{T}\hat{B}_-}\{\hat{z}_2(t)\,\mathcal{G}_{CC}^{(+)}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1) - \hat{y}_2(t)\,\mathcal{G}_{SC}^{(+)}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1)\}\sqrt{\hat{H}_2\hat{T}\hat{B}_-} + \frac{\gamma}{2}\hat{\nu}_1^{-1}\hat{H}_2^{3/4}\{\hat{z}_1(t)\,\mathcal{G}_{SS}^{(-)}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2) + \hat{y}_1(t)\,\mathcal{G}_{CS}^{(-)}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2)\}\hat{H}_1^{1/4} \tag{5.8b}$$

$$\hat{\sigma}_{21}^P(t) = \frac{\gamma}{2}\hat{\nu}_2^{-1}\sqrt{\hat{B}_+\hat{T}^\dagger\hat{H}_2}\{\hat{z}_1(t)\,\mathcal{G}_{CC}^{(+)}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2) - \hat{y}_1(t)\,\mathcal{G}_{SC}^{(+)}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2)\}\sqrt{\hat{B}_+\hat{T}^\dagger} + \frac{\gamma}{2}\hat{\nu}_2^{-1}\hat{H}_1^{1/4}\{\hat{z}_2(t)\,\mathcal{G}_{SS}^{(-)}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1) - \hat{y}_2(t)\,\mathcal{G}_{CS}^{(-)}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1)\}\hat{H}_2^{3/4} \tag{5.8c}$$

$$\hat{\sigma}_{22}^P(t) = \mathrm{i}\frac{\gamma}{2}\hat{\nu}_2^{-1}\sqrt{\hat{B}_+\hat{T}^\dagger\hat{H}_2}\{\hat{z}_1(t)\,\mathcal{G}_{CS}^{(+)}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2) - \hat{y}_1(t)\,\mathcal{G}_{SS}^{(+)}(t;\hat{\nu}_1,\hat{\omega}_1,\hat{\omega}_2)\}\hat{H}_1^{1/4} + \mathrm{i}\frac{\gamma}{2}\hat{\nu}_2^{-1}\hat{H}_1^{1/4}\{\hat{z}_2(t)\,\mathcal{G}_{SC}^{(-)}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1) + \hat{y}_2(t)\,\mathcal{G}_{CC}^{(-)}(t;\hat{\nu}_2,\hat{\omega}_2,\hat{\omega}_1)\}\sqrt{\hat{H}_2\hat{T}\hat{B}_-}. \tag{5.8d}$$

The auxiliary functions, $\mathcal{G}_{XY}^{(\pm)}(t;\hat{p},\hat{q},\hat{r})$, are given by equations (3.32) and (3.33). From equations (3.34) and (3.35), we have for the elements of the population inversion matrix:

$$[\hat{\sigma}_3(t)]_{ij} = \cos(\hat{\nu}_i t)\,[\hat{\sigma}_3(0)]_{ij} + \frac{2\mathrm{i}\alpha}{\hbar}\,\sin(\hat{\nu}_i t)\,\hat{\nu}_i^{-1}[\hat{\boldsymbol{S}}_i(0)\,\hat{\sigma}_3(0)]_{ij} + \hat{\sigma}_{ij}^P(t). \tag{5.9}$$

5.1. The intensity-dependent resonant limit

In this limit we set ($\Delta = 0$), so the evolution matrix of the system is given by

$$\hat{\boldsymbol{U}}_i(t,0) = \begin{bmatrix} \cos\left(\frac{1}{2}\hat{\nu}_1 t\right) & \sin\left(\frac{1}{2}\hat{\nu}_1 t\right)\hat{C} \\ -\sin\left(\frac{1}{2}\hat{\nu}_2 t\right)\hat{C}^\dagger & \cos\left(\frac{1}{2}\hat{\nu}_2 t\right) \end{bmatrix} \tag{5.10}$$

and the elements of the population inversion factor can be written as

$$[\hat{\sigma}_3(t)]_{ij} = \cos(\hat{\nu}_i t)\,[\hat{\sigma}_3(0)]_{ij} + \frac{2\mathrm{i}\alpha}{\hbar}\,\sin(\hat{\nu}_i t)\,\hat{\nu}_i^{-1}[\hat{\boldsymbol{S}}_i(0)\,\hat{\sigma}_3(0)]_{ij}. \tag{5.11}$$

5.2. The standard intensity-dependent Jaynes–Cummings limit

For the case of a harmonic oscillator system ($\hat{T} = \hat{T}^\dagger \longrightarrow 1$, $\hat{B}_- \longrightarrow \hat{a}$, $\hat{B}_+ \longrightarrow \hat{a}^\dagger$ and $[\hat{a},\hat{a}^\dagger] = \hbar\omega$), we have for the $\hat{\sigma}_{ij}^P(t)$ matrix elements the following expressions:

$$\hat{\sigma}_{11}^P(t) = \mathrm{i}\frac{\gamma}{2}\hat{\nu}_1^{-1}\sqrt{\hat{a}}\{\mathcal{K}_S(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_2) - \mathcal{K}_S(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_2)\}(\hat{a}\hat{a}^\dagger)^{3/4}$$

$$-\mathrm{i}\frac{\gamma}{2}\hat{\nu}_1^{-1}(\hat{a}\hat{a}^\dagger)^{3/4}\{\mathcal{K}_S(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_1)-\mathcal{K}_S(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_1)\}\sqrt{\hat{a}^\dagger} \qquad (5.12a)$$

$$\hat{\sigma}_{12}^P(t)=\frac{\gamma}{2}\hat{\nu}_1^{-1}\sqrt{\hat{a}}\{\mathcal{K}_C(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_2)-\mathcal{K}_C(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_2)\}\sqrt{\hat{a}\hat{a}^\dagger\hat{a}}$$

$$-\frac{\gamma}{2}\hat{\nu}_1^{-1}(\hat{a}\hat{a}^\dagger)^{3/4}\{\mathcal{K}_C(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_1)-\mathcal{K}_C(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_1)\}(\hat{a}^\dagger\hat{a})^{1/4} \qquad (5.12b)$$

$$\hat{\sigma}_{21}^P(t)=\frac{\gamma}{2}\hat{\nu}_2^{-1}\sqrt{\hat{a}^\dagger\hat{a}\hat{a}^\dagger}\{\mathcal{K}_C(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_1)+\mathcal{K}_C(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_1)\}\sqrt{\hat{a}^\dagger}$$

$$-\frac{\gamma}{2}\hat{\nu}_2^{-1}(\hat{a}^\dagger\hat{a})^{1/4}\{\mathcal{K}_C(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_2)-\mathcal{K}_C(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_2)\}(\hat{a}\hat{a}^\dagger)^{3/4} \qquad (5.12c)$$

$$\hat{\sigma}_{22}^P(t)=\mathrm{i}\frac{\gamma}{2}\hat{\nu}_2^{-1}\sqrt{\hat{a}^\dagger\hat{a}\hat{a}^\dagger}\{\mathcal{K}_S(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_1)+\mathcal{K}_S(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_1)\}(\hat{a}^\dagger\hat{a})^{1/4}$$

$$-\mathrm{i}\frac{\gamma}{2}\hat{\nu}_2^{-1}(\hat{a}^\dagger\hat{a})^{1/4}\{\mathcal{K}_S(t;\hat{\omega}_2,\hat{\omega}_1,\hat{\nu}_2)+\mathcal{K}_S(t;\hat{\omega}_2,-\hat{\omega}_1,\hat{\nu}_2)\}\sqrt{\hat{a}\hat{a}^\dagger\hat{a}} \qquad (5.12d)$$

where the auxiliary functions, $\mathcal{K}_S(t;\hat{p},\hat{q},\hat{r})$ and $\mathcal{K}_C(t;\hat{p},\hat{q},\hat{r})$, are given by equations (3.40). The final expression for the population inversion factor has the same form given by equation (5.9), with

$$\hbar\hat{\nu}_1=2\alpha\,\hat{a}\hat{a}^\dagger \qquad \hbar\hat{\nu}_2=2\alpha\,\hat{a}^\dagger\hat{a} \qquad (5.13a)$$

$$\hbar\hat{\omega}_1=\alpha\sqrt{(\hat{a}\hat{a}^\dagger)^2+\beta^2} \qquad \hbar\hat{\omega}_2=\alpha\sqrt{(\hat{a}^\dagger\hat{a})^2+\beta^2}. \qquad (5.13b)$$

6. Conclusions

In this paper we extended our earlier work [8] on bound-state problems which represent two-level systems. The corresponding coupled-channel Hamiltonians generalize the nonresonant and intensity-dependent nonresonant Jaynes–Cummings Hamiltonians. In the case of a nonresonant system, if we take the starting Hamiltonian to be the simplest shape-invariant system, namely the harmonic oscillator, our results reduce to those of the standard nonresonant Jaynes–Cummings approach, which has been extensively used to model a two-level atom single-field mode interaction whose detuning is not null. In addition we have studied time evolution and population inversion factors of both kinds of generalized systems.

These models are not only interesting on their own account. Being exactly solvable coupled-channels problems they may help to assess the validity and accuracy of various approximate approaches to the coupled-channel problems [19].

Acknowledgments

This paper was supported in part by the US National Science Foundation grants no PHY-9605140 and PHY-0070161 at the University of Wisconsin, and in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation. ABB acknowledges the support of the Alexander von Humboldt-Stiftung. MACR acknowledges the support of Fundação de Amparoà Pesquisa do Estado de São Paulo (contract no 98/13722-2). ANFA acknowledges the support of Fundação Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (contract no BEX0610/96-8). ABB is grateful to the Max-Planck-Institut für Kernphysik and H A Weidenmüller for their very kind hospitality.

Appendix A

Here we give the steps used to obtain the specific form of the operators $\hat{C}$ and $\hat{D}$. Using equation (3.10) in the unitary condition equation (3.11) we can show that the $\hat{C}$ and $\hat{D}$ operators need to satisfy the following six conditions:

$$\hat{C}\hat{C}^\dagger = \hat{C}^\dagger\hat{C} = 1 \tag{A.1a}$$

$$\hat{D}\hat{D}^\dagger = \hat{D}^\dagger\hat{D} = 1 \tag{A.1b}$$

$$\hat{D}^\dagger \sin(\hat{\omega}_2 t) = -\sin(\hat{\omega}_1 t)\,\hat{C} \tag{A.1c}$$

$$\hat{D} \cos(\hat{\omega}_1 t) = -\cos(\hat{\omega}_2 t)\,\hat{C}^\dagger. \tag{A.1d}$$

At this point we can use the following property:

$$\begin{aligned}\sqrt{\hat{T}\hat{B}_-}\,\hat{\omega}_2 &= \sqrt{\hat{T}\hat{B}_-}\,\sqrt{\alpha^2\hat{B}_+\hat{B}_- + \beta^2}/\hbar \\ &= \sqrt{\alpha^2\hat{T}\hat{B}_-\hat{B}_+\hat{B}_- + \hat{T}\hat{B}_-\beta^2}/\hbar \\ &= \sqrt{\alpha^2\hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger\hat{T}\hat{B}_- + \beta^2\hat{T}\hat{B}_-}/\ hbar \\ &= \sqrt{\alpha^2\hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger + \beta^2}/\hbar\,\sqrt{\hat{T}\hat{B}_-} \\ &= \hat{\omega}_1\sqrt{\hat{T}\hat{B}_-}.\end{aligned} \tag{A.2}$$

Then, with this result we have

$$\sqrt{\hat{T}\hat{B}_-}\,\hat{\omega}_2^2 = \sqrt{\hat{T}\hat{B}_-}\,\hat{\omega}_2\,\hat{\omega}_2 = \hat{\omega}_1\sqrt{\hat{T}\hat{B}_-}\,\hat{\omega}_2 = \hat{\omega}_1^2\sqrt{\hat{T}\hat{B}_-} \tag{A.3}$$

and finally, by induction, we conclude that

$$\sqrt{\hat{T}\hat{B}_-}\,\hat{\omega}_2^n = \hat{\omega}_1^n\sqrt{\hat{T}\hat{B}_-}. \tag{A.4}$$

In the same way,

$$\begin{aligned}\sqrt{\hat{B}_+\hat{T}^\dagger}\,\hat{\omega}_1 &= \sqrt{\hat{B}_+\hat{T}^\dagger}\,\sqrt{\alpha^2\hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger + \beta^2}/\hbar \\ &= \sqrt{\alpha^2\hat{B}_+\hat{T}^\dagger\hat{T}\hat{B}_-\hat{B}_+\hat{T}^\dagger + \hat{B}_+\hat{T}^\dagger\beta^2}/\hbar \\ &= \sqrt{\alpha^2\hat{B}_+\hat{B}_-\hat{B}_+\hat{T}^\dagger + \beta^2\hat{B}_+\hat{T}^\dagger}/\hbar \\ &= \sqrt{\alpha^2\hat{B}_+\hat{B}_- + \beta^2}/\hbar\,\sqrt{\hat{B}_+\hat{T}^\dagger} \\ &= \hat{\omega}_2\sqrt{\hat{B}_+\hat{T}^\dagger}.\end{aligned} \tag{A.5}$$

Then, with this result we have

$$\sqrt{\hat{B}_+\hat{T}^\dagger}\,\hat{\omega}_1^2 = \sqrt{\hat{B}_+\hat{T}^\dagger}\,\hat{\omega}_1\,\hat{\omega}_1 = \hat{\omega}_2\sqrt{\hat{B}_+\hat{T}^\dagger}\,\hat{\omega}_1 = \hat{\omega}_2^2\sqrt{\hat{B}_+\hat{T}^\dagger} \tag{A.6}$$

and finally, again by induction, we get

$$\sqrt{\hat{B}_+\hat{T}^\dagger}\,\hat{\omega}_1^n = \hat{\omega}_2^n\sqrt{\hat{B}_+\hat{T}^\dagger}. \tag{A.7}$$

Using the properties given by equations (A.4) and (A.7) and the forms of the $\hat{C}$ and $\hat{D}$ operators, defined by equations (3.12), we can verify that

$$\hat{C}\hat{C}^\dagger = \hat{D}^\dagger\hat{D} = \frac{\mathrm{i}}{\hat{H}_2^{1/4}}\sqrt{\hat{T}\hat{B}_-}\,\sqrt{\hat{B}_+\hat{T}^\dagger}\frac{(-\mathrm{i})}{\hat{H}_2^{1/4}} = \frac{1}{\hat{H}_2^{1/4}}\sqrt{\hat{H}_2}\frac{1}{\hat{H}_2^{1/4}} = 1 \tag{A.8}$$

and

$$\hat{C}^{\dagger}\hat{C} = \hat{D}\hat{D}^{\dagger} = \sqrt{\hat{B}_{+}\hat{T}^{\dagger}}\frac{(-\mathrm{i})}{\hat{H}_2^{1/4}}\,\frac{\mathrm{i}}{\hat{H}_2^{1/4}}\sqrt{\hat{T}\hat{B}_{-}} = \sqrt{\hat{B}_{+}\hat{T}^{\dagger}}\frac{1}{\sqrt{\hat{H}_2}}\sqrt{\hat{H}_2}\frac{1}{\sqrt{\hat{B}_{+}\hat{T}^{\dagger}}} = 1. \tag{A.9}$$

Also, using the series expansion of the trigonometric functions, we can show that

$$\begin{aligned}
\hat{D}^{\dagger}\,\sin(\hat{\omega}_2 t) &= \frac{-\mathrm{i}}{\hat{H}_2^{1/4}}\sqrt{\hat{T}\hat{B}_{-}}\sum_{n=0}^{\infty}(-1)^n\frac{(\hat{\omega}_2 t)^{2n+1}}{(2n+1)!}\\
&= \frac{-\mathrm{i}}{\hat{H}_2^{1/4}}\sum_{n=0}^{\infty}(-1)^n\sqrt{\hat{T}\hat{B}_{-}}\frac{(\hat{\omega}_2 t)^{2n+1}}{(2n+1)!}\\
&= \frac{-\mathrm{i}}{\hat{H}_2^{1/4}}\sum_{n=0}^{\infty}(-1)^n\frac{(\hat{\omega}_1 t)^{2n+1}}{(2n+1)!}\sqrt{\hat{T}\hat{B}_{-}}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(\hat{\omega}_1 t)^{2n+1}}{(2n+1)!}\,\frac{-\mathrm{i}}{\hat{H}_2^{1/4}}\sqrt{\hat{T}\hat{B}_{-}}\\
&= -\sin(\hat{\omega}_1 t)\,\hat{C}
\end{aligned} \tag{A.10}$$

where we used the commutation between $\hat{H}_2$ and $\hat{\omega}_1$ (see appendix B). In the same way we can prove that

$$\begin{aligned}
\hat{D}\,\cos(\hat{\omega}_1 t) &= \sqrt{\hat{B}_{+}\hat{T}^{\dagger}}\frac{\mathrm{i}}{\hat{H}_2^{1/4}}\sum_{n=0}^{\infty}(-1)^n\frac{(\hat{\omega}_1 t)^{2n}}{(2n)!}\\
&= \sum_{n=0}^{\infty}(-1)^n\sqrt{\hat{B}_{+}\hat{T}^{\dagger}}\frac{(\hat{\omega}_1 t)^{2n}}{(2n)!}\,\frac{\mathrm{i}}{\hat{H}_2^{1/4}}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(\hat{\omega}_2 t)^{2n}}{(2n)!}\sqrt{\hat{B}_{+}\hat{T}^{\dagger}}\frac{\mathrm{i}}{\hat{H}_2^{1/4}}\\
&= -\cos(\hat{\omega}_2 t)\,\hat{C}^{\dagger}.
\end{aligned} \tag{A.11}$$

Again, we used the commutation between $\hat{H}_2$ and $\hat{\omega}_1$.

Appendix B

In this appendix we show the necessary steps to obtain the explicit expressions of the particular solution elements of the population inversion factor. To resolve the integrals in equation (3.30), first we need to determine the elements of the $\hat{\boldsymbol{F}}(t)$ matrix. To do that we can use equations (2.2), (3.13) and (3.23) to write down

$$\begin{aligned}
\hat{F}_{11}(t) &= -\gamma\{\cos(\hat{\omega}_1 t)\,\hat{T}\hat{B}_{-}\,\sin(\hat{\omega}_2 t)\,\hat{C}^{\dagger} + \hat{C}\,\sin(\hat{\omega}_2 t)\,\hat{B}_{+}\hat{T}^{\dagger}\,\cos(\hat{\omega}_1 t)\}\\
&= \mathrm{i}\gamma\left\{\sqrt{\hat{T}\hat{B}_{-}}\,\cos(\hat{\omega}_2 t)\,\sin(\hat{\omega}_1 t)\,\hat{H}_2^{1/4} - \hat{H}_2^{1/4}\,\sin(\hat{\omega}_1 t)\,\cos(\hat{\omega}_2 t)\sqrt{\hat{B}_{+}\hat{T}^{\dagger}}\right\}
\end{aligned} \tag{B.1a}$$

$$\begin{aligned}
\hat{F}_{12}(t) &= \gamma\{\cos(\hat{\omega}_1 t)\,\hat{T}\hat{B}_{-}\,\cos(\hat{\omega}_2 t) - \hat{C}\,\sin(\hat{\omega}_2 t)\,\hat{B}_{+}\hat{T}^{\dagger}\,\sin(\hat{\omega}_1 t)\,\hat{C}\}\\
&= \gamma\left\{\sqrt{\hat{T}\hat{B}_{-}}\,\cos(\hat{\omega}_2 t)\,\cos(\hat{\omega}_1 t)\sqrt{\hat{T}\hat{B}_{-}} + \hat{H}_2^{1/4}\,\sin(\hat{\omega}_1 t)\,\sin(\hat{\omega}_2 t)\,\hat{H}_1^{1/4}\right\}
\end{aligned} \tag{B.1b}$$

$$\hat{F}_{21}(t) = \gamma\{\cos(\hat{\omega}_2 t)\,\hat{B}_+\hat{T}^\dagger\,\cos(\hat{\omega}_1 t) - \hat{C}^\dagger\,\sin(\hat{\omega}_1 t)\,\hat{T}\hat{B}_-\,\sin(\hat{\omega}_2 t)\,\hat{C}^\dagger\}$$
$$= \gamma\left\{\sqrt{\hat{B}_+\hat{T}^\dagger}\,\cos(\hat{\omega}_1 t)\,\cos(\hat{\omega}_2 t)\sqrt{\hat{B}_+\hat{T}^\dagger} + \hat{H}_1^{1/4}\,\sin(\hat{\omega}_2 t)\,\sin(\hat{\omega}_1 t)\,\hat{H}_2^{1/4}\right\} \quad \text{(B.1}c\text{)}$$

$$\hat{F}_{22}(t) = \gamma\{\hat{C}^\dagger\,\sin(\hat{\omega}_1 t)\,\hat{T}\hat{B}_-\,\cos(\hat{\omega}_2 t) + \cos(\hat{\omega}_2 t)\,\hat{B}_+\hat{T}^\dagger\,\sin(\hat{\omega}_1 t)\,\hat{C}\}$$
$$= \mathrm{i}\gamma\left\{\sqrt{\hat{B}_+\hat{T}^\dagger}\,\cos(\hat{\omega}_1 t)\,\sin(\hat{\omega}_2 t)\,\hat{H}_1^{1/4} - \hat{H}_1^{1/4}\,\sin(\hat{\omega}_2 t)\,\cos(\hat{\omega}_1 t)\sqrt{\hat{T}\hat{B}_-}\right\} \quad \text{(B.1}d\text{)}$$

where $\gamma = 4\alpha^2\beta/\hbar^2$. Here we used the properties (A.1), (A.4) and (A.7), together with the following operators relations:

$$\hat{C}\sqrt{\hat{B}_+\hat{T}^\dagger} = -\sqrt{\hat{T}\hat{B}_-}\,\hat{C}^\dagger = \mathrm{i}\hat{H}_2^{1/4} \quad \text{(B.2}a\text{)}$$
$$\sqrt{\hat{B}_+\hat{T}^\dagger}\,\hat{C} = -\hat{C}^\dagger\sqrt{\hat{T}\hat{B}_-} = \mathrm{i}\hat{H}_1^{1/4}. \quad \text{(B.2}b\text{)}$$

Now, keeping in mind that $[\hat{\nu}_j, \hat{\omega}_j] = 0$, ($j = 1$, or 2), we may use the trigonometric relationships involving products of trigonometric functions with arguments $\hat{\nu}_j t$ and $\hat{\omega}_j t$ (since we have $\exp(\hat{\nu}_j t)\,\exp(\pm\hat{\omega}_j t) = \exp[(\hat{\nu}_j \pm \hat{\omega}_j)t]$). Then, using those relationships, the following commutators:

$$[\hat{\nu}_1, \hat{H}_2] = [\hat{\omega}_1, \hat{H}_2] = [\hat{\nu}_2, \hat{H}_1] = [\hat{\omega}_2, \hat{H}_1] = 0 \quad \text{(B.3)}$$

and the same properties (1), (A.4) and (A.7), we can show that

$$\hat{y}_1(t)\,\hat{F}_{11}(t) = \mathrm{i}\frac{\gamma}{2}\sqrt{\hat{T}\hat{B}_-}\{\cos[(\hat{\nu}_2 - \hat{\omega}_2)t]\,\sin(\hat{\omega}_1 t) + \cos[(\hat{\nu}_2 + \hat{\omega}_2)t]\,\sin(\hat{\omega}_1 t)\}\hat{H}_2^{1/4}$$
$$+\mathrm{i}\frac{\gamma}{2}\hat{H}_2^{1/4}\{\sin[(\hat{\nu}_1 - \hat{\omega}_1)t]\,\cos(\hat{\omega}_2 t) - \sin[(\hat{\nu}_1 + \hat{\omega}_1)t]\,\cos(\hat{\omega}_2 t)\}\sqrt{\hat{B}_+\hat{T}^\dagger} \quad \text{(A.4}a\text{)}$$

$$\hat{y}_1(t)\,\hat{F}_{12}(t) = \frac{\gamma}{2}\sqrt{\hat{T}\hat{B}_-}\{\cos[(\hat{\nu}_2 - \hat{\omega}_2)t]\,\cos(\hat{\omega}_1 t) + \cos[(\hat{\nu}_2 + \hat{\omega}_2)t]\,\cos(\hat{\omega}_1 t)\}\sqrt{\hat{T}\hat{B}_-}$$
$$+\frac{\gamma}{2}\hat{H}_2^{1/4}\{\sin[(\hat{\nu}_1 + \hat{\omega}_1)t]\,\sin(\hat{\omega}_2 t) - \sin[(\hat{\nu}_1 - \hat{\omega}_1)t]\,\sin(\hat{\omega}_2 t)\}\hat{H}_1^{1/4} \quad \text{(A.4}b\text{)}$$

$$\hat{y}_2(t)\,\hat{F}_{21}(t) = \frac{\gamma}{2}\sqrt{\hat{B}_+\hat{T}^\dagger}\{\cos[(\hat{\nu}_1 - \hat{\omega}_1)t]\,\cos(\hat{\omega}_2 t) + \cos[(\hat{\nu}_1 + \hat{\omega}_1)t]\,\cos(\hat{\omega}_2 t)\}\sqrt{\hat{B}_+\hat{T}^\dagger}$$
$$+\frac{\gamma}{2}\hat{H}_1^{1/4}\{\sin[(\hat{\nu}_2 + \hat{\omega}_2)t]\,\sin(\hat{\omega}_1 t) - \sin[(\hat{\nu}_2 - \hat{\omega}_2)t]\,\sin(\hat{\omega}_1 t)\}\hat{H}_2^{1/4} \quad \text{(A.4}c\text{)}$$

$$\hat{y}_2(t)\,\hat{F}_{22}(t) = \mathrm{i}\frac{\gamma}{2}\sqrt{\hat{B}_+\hat{T}^\dagger}\{\cos[(\hat{\nu}_1 - \hat{\omega}_1)t]\,\sin(\hat{\omega}_2 t) + \cos[(\hat{\nu}_1 + \hat{\omega}_1)t]\,\sin(\hat{\omega}_2 t)\}\hat{H}_1^{1/4}$$
$$+\mathrm{i}\frac{\gamma}{2}\hat{H}_1^{1/4}\{\sin[(\hat{\nu}_2 - \hat{\omega}_2)t]\,\cos(\hat{\omega}_1 t) - \sin[(\hat{\nu}_2 + \hat{\omega}_2)t]\,\cos(\hat{\omega}_1 t)\}\sqrt{\hat{T}\hat{B}_-}. \quad \text{(A.4}d\text{)}$$

In a similar way, we can show that

$$\hat{z}_1(t)\,\hat{F}_{11}(t) = \mathrm{i}\frac{\gamma}{2}\sqrt{\hat{T}\hat{B}_-}\{\sin[(\hat{\nu}_2 - \hat{\omega}_2)t]\,\sin(\hat{\omega}_1 t) + \sin[(\hat{\nu}_2 + \hat{\omega}_2)t]\,\sin(\hat{\omega}_1 t)\}\hat{H}_2^{1/4}$$
$$-\mathrm{i}\frac{\gamma}{2}\hat{H}_2^{1/4}\{\cos[(\hat{\nu}_1 - \hat{\omega}_1)t]\,\cos(\hat{\omega}_2 t) - \cos[(\hat{\nu}_1 + \hat{\omega}_1)t]\,\cos(\hat{\omega}_2 t)\}\sqrt{\hat{B}_+\hat{T}^\dagger} \quad \text{(B.5}a\text{)}$$

$$\hat{z}_1(t)\,\hat{F}_{12}(t) = \frac{\gamma}{2}\sqrt{\hat{T}\hat{B}_-}\{\sin[(\hat{\nu}_2-\hat{\omega}_2)t]\,\cos(\hat{\omega}_1 t)+\sin[(\hat{\nu}_2+\hat{\omega}_2)t]\,\cos(\hat{\omega}_1 t)\}\sqrt{\hat{T}\hat{B}_-}$$
$$-\frac{\gamma}{2}\hat{H}_2^{1/4}\{\cos[(\hat{\nu}_1+\hat{\omega}_1)t]\,\sin(\hat{\omega}_2 t)-\cos[(\hat{\nu}_1-\hat{\omega}_1)t]\,\sin(\hat{\omega}_2 t)\}\hat{H}_1^{1/4} \quad \text{(B.5}b\text{)}$$

$$\hat{z}_2(t)\,\hat{F}_{21}(t) = \frac{\gamma}{2}\sqrt{\hat{B}_+\hat{T}^\dagger}\{\sin[(\hat{\nu}_1-\hat{\omega}_1)t]\,\cos(\hat{\omega}_2 t)+\sin[(\hat{\nu}_1+\hat{\omega}_1)t]\,\cos(\hat{\omega}_2 t)\}\sqrt{\hat{B}_+\hat{T}^\dagger}$$
$$-\frac{\gamma}{2}\hat{H}_1^{1/4}\{\cos[(\hat{\nu}_2+\hat{\omega}_2)t]\,\sin(\hat{\omega}_1 t)-\cos[(\hat{\nu}_2-\hat{\omega}_2)t]\,\sin(\hat{\omega}_1 t)\}\hat{H}_2^{1/4} \quad \text{(B.5}c\text{)}$$

$$\hat{z}_2(t)\,\hat{F}_{22}(t) = \mathrm{i}\frac{\gamma}{2}\sqrt{\hat{B}_+\hat{T}^\dagger}\{\sin[(\hat{\nu}_1-\hat{\omega}_1)t]\,\sin(\hat{\omega}_2 t)+\sin[(\hat{\nu}_1+\hat{\omega}_1)t]\,\sin(\hat{\omega}_2 t)\}\hat{H}_1^{1/4}$$
$$-\mathrm{i}\frac{\gamma}{2}\hat{H}_1^{1/4}\{\cos[(\hat{\nu}_2-\hat{\omega}_2)t]\,\cos(\hat{\omega}_1 t)-\cos[(\hat{\nu}_2+\hat{\omega}_2)t]\,\cos(\hat{\omega}_1 t)\}\sqrt{\hat{T}\hat{B}_-}. \quad \text{(B.5}d\text{)}$$

The non-commutativity between the operators $\hat{\omega}_1$ and $\hat{\omega}_2$ implies that to calculate the integrals involving the terms given by equations (B.4) and (B.5) we need to use the series expansion of the trigonometric functions. In this case the integrals can be easily done because the time variable can be considered as a parameter factor. Finally, using these results in equation (3.30) is trivial to find the expression (3.31) for the matrix elements of the particular solution.

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